su(*N*) tensor product multiplicities and virtual Berenstein–Zelevinsky triangles

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Abstract
Information on $su(N)$ tensor product multiplicities is neatly encoded in
Berenstein–Zelevinsky triangles. Here we study a generalization of these
triangles by allowing negative as well as non-negative integer entries. For
a fixed triple product of weights, these generalized Berenstein–Zelevinsky
triangles span a lattice in which one may move by adding integer linear
combinations of so-called virtual triangles. Inequalities satisfied by the
coefficients of the virtual triangles describe a polytope. The tensor product
multiplicities may be computed as the number of integer points in this convex
polytope. As our main result, we present an explicit formula for this discretised
volume as a multiple sum. As an application, we also address the problem of
determining when a tensor product multiplicity is non-vanishing. The solution
is represented by a set of inequalities in the Dynkin labels. We also allude
to the question of when a tensor product multiplicity is greater than a given
non-negative integer.

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1. Introduction

The decomposition of the tensor products of modules of simple Lie algebras has been studied
for a long time now. Many elegant results have been found for the multiplicities of the
decompositions, the so-called tensor product multiplicities. The relatively recent Berenstein–
Zelevinsky method of triangles [1] is an example. Although it is a powerful, symmetric
method, it is not explicit: triangles are constructed according to certain rules, and their
number is the required tensor product multiplicity. Here we show that a generalization of
these Berenstein–Zelevinsky (BZ) triangles allows us to work out a very explicit expression
for the multiplicities. A tensor product multiplicity is expressed as a multiple sum, counting
the number of integer points in a particular convex polytope, to be defined below. BZ triangles

and our results pertain to the A-series; $A_r = su(r + 1)$. We will sometimes write $su(N)$ with $N = r + 1$.

We are interested in describing decompositions of tensor products of irreducible highest weight modules of simple Lie algebras. They are usually written

$$M_\lambda \otimes M_\mu = \bigoplus_\nu T^\nu_{\lambda,\mu} M_\nu \tag{1}$$

where M_λ is the module of highest weight λ and $T^\nu_{\lambda,\mu}$ is the tensor product multiplicity. We study the equivalent but more symmetric problem of determining the multiplicity of the singlet in the expansion of the triple product

$$M_\lambda \otimes M_\mu \otimes M_\nu \supset T_{\lambda,\mu,\nu} M_0. \tag{2}$$

If ν^+ denotes the highest weight conjugate to ν, we have $T_{\lambda,\mu,\nu} = T^{\nu^+}_{\lambda,\mu}$. We will use the shorthand notation $\lambda \otimes \mu \otimes \nu$ to represent the left-hand side of (2).

An $su(3)$ BZ triangle, describing a particular coupling (to the singlet) associated with the triple product $\lambda \otimes \mu \otimes \nu$, is a triangular arrangement of nine non-negative integers:

$$
\begin{array}{ccccccc}
 & & & m_{13} & & & \\
 & & n_{12} & & l_{23} & & \\
 & m_{23} & & & & m_{12} & \\
n_{13} & & l_{12} & & n_{23} & & l_{13}
\end{array}
\tag{3}
$$

These integers are related to the Dynkin labels of the three integrable highest weights by

$$
\begin{array}{lll}
m_{13} + n_{12} = \lambda_1 & n_{13} + l_{12} = \mu_1 & l_{13} + m_{12} = \nu_1 \\
m_{23} + n_{13} = \lambda_2 & n_{23} + l_{13} = \mu_2 & l_{23} + m_{13} = \nu_2.
\end{array}
\tag{4}
$$

We call these relations outer constraints. The entries further satisfy the so-called hexagon conditions

$$
\begin{aligned}
n_{12} + m_{23} &= n_{23} + m_{12} \\
m_{12} + l_{23} &= m_{23} + l_{12} \\
l_{12} + n_{23} &= l_{23} + n_{12}
\end{aligned}
\tag{5}
$$

of which only two are independent. They say that the lengths of opposite sides of the hexagon must be equal, if the length of a segment is defined to be the sum of the two integers associated with its endpoints. An $su(3)$ BZ triangle is thus composed of one hexagon and three corner points.

For $su(4)$ the BZ triangle is defined in a similar way, in terms of 18 non-negative integers:

$$
\begin{array}{ccccccccccc}
 & & & & & m_{14} & & & & & \\
 & & & & n_{12} & & l_{34} & & & & \\
 & & & m_{24} & & & & m_{13} & & & \\
 & & n_{13} & & l_{23} & & n_{23} & & l_{24} & & \\
 & m_{34} & & & & m_{23} & & & & m_{12} & \\
n_{14} & & l_{12} & & n_{24} & & l_{13} & & n_{34} & & l_{14}
\end{array}
\tag{6}
$$

related to the Dynkin labels by

$$
\begin{array}{lll}
m_{14} + n_{12} = \lambda_1 & n_{14} + l_{12} = \mu_1 & l_{14} + m_{12} = \nu_1 \\
m_{24} + n_{13} = \lambda_2 & n_{24} + l_{13} = \mu_2 & l_{24} + m_{13} = \nu_2 \\
m_{34} + n_{14} = \lambda_3 & n_{34} + l_{14} = \mu_3 & l_{34} + m_{14} = \nu_3.
\end{array}
\tag{7}
$$

Furthermore, the *su*(4) BZ triangle contains three hexagons:

$$
\begin{aligned}
n_{12} + m_{24} &= m_{13} + n_{23} & n_{13} + l_{23} &= l_{12} + n_{24} & l_{24} + n_{23} &= l_{13} + n_{34} \\
n_{12} + l_{34} &= l_{23} + n_{23} & n_{13} + m_{34} &= n_{24} + m_{23} & n_{23} + m_{23} &= m_{12} + n_{34} \\
m_{24} + l_{23} &= l_{34} + m_{13} & m_{34} + l_{12} &= l_{23} + m_{23} & l_{13} + m_{23} &= l_{24} + m_{12}.
\end{aligned} \tag{8}
$$

It is a general feature for any N that only two out of the three hexagon identities associated with a particular hexagon are independent.

The $su(N)$ generalization is obvious; the triangle is built out of $(N-1)(N-2)/2$ hexagons and three corner points. Simple examples of lower rank BZ triangles and their applications may be found in [2].

2. Generalised and virtual Berenstein–Zelevinsky triangles

The generalization of the BZ triangles that we consider is obtained by weakening the constraint that all entries are *non-negative* integers to *arbitrary* integers, negative as well as non-negative. The hexagon identities and the outer constraints are still enforced. A triangle will be called a *true* BZ triangle if all its entries are non-negative.

We consider a generalized BZ triangle associated with $su(r + 1)$. Denoting the number of entries E_r and the number of hexagons H_r, we have

$$
E_r = \tfrac{3}{2}r(r + 1) \qquad H_r = \tfrac{1}{2}r(r - 1). \tag{9}
$$

For a given triple product $\lambda \otimes \mu \otimes \nu$, the set of associated triangles spans an H_r-dimensional lattice. Each hexagon corresponds to two independent constraints on the triangle entries while there are $3r$ outer constraints. This leaves

$$
E_r - (2H_r + 3r) = H_r \tag{10}
$$

parameters labelling the possible triangles. Among these, only a finite number are true BZ triangles. This number is precisely the tensor product multiplicity of the triple coupling. For example, when the singlet does not occur in the decomposition of the triple product, there are no true BZ triangles in the lattice.

A special class of generalized BZ triangles is associated with the triple product $0 \otimes 0 \otimes 0$. We say they have weight $(\lambda, \mu, \nu) = (0, 0, 0)$. According to the general argument above, $H_r = \tfrac{1}{2}r(r - 1)$ such triangles are linearly independent. We call them *virtual* triangles and denote them using $\mathcal{V}$. It is natural to exclude the triangle with all entries equal to zero from the set of virtual triangles. It is the unique true BZ triangle in the lattice associated with the triple product $0 \otimes 0 \otimes 0$. In the cases of $su(3)$ and $su(4)$, virtual triangles have appeared in [3] and [4], respectively.

A convenient basis for the virtual triangles is given by associating a simple distribution of plus and minus ones (written 1 and $\bar{1} \equiv -1$, respectively) with a given hexagon. All other entries are zero. The distribution is

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & 1 & \bar{1} & & \bar{1} & 1 & \\
 & & \bar{1} & & \bar{1} & & \\
 & 1 & \bar{1} & & \bar{1} & 1 & \\
 & & & 1 & & &
\end{array} \tag{11}
$$

Thus, a basis virtual triangle will always have six entries equalling -1, and between three and six entries equalling $+1$. The number of $+1$ entries depends on where the associated hexagon

is situated in the generalized BZ triangle and the rank of $su(r + 1)$ (three entries equal +1 for $su(3)$ only). That these virtual triangles are indeed linearly independent is obvious.

There are no virtual triangles in the case of $su(2)$. In the case of $su(3)$ there is one basis virtual triangle

$$
\mathcal{V} = \begin{matrix}
 & & 1 & & \\
 & \bar{1} & & \bar{1} & \\
 \bar{1} & & & & \bar{1} \\
 1 & & \bar{1} & \bar{1} & & 1
\end{matrix}
\tag{12}
$$

while in the case of $su(4)$ the three basis virtual triangles $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are

$$
\mathcal{V}_1 = \begin{matrix}
 & & & 1 & & & \\
 & & \bar{1} & & \bar{1} & & \\
 & \bar{1} & & & & \bar{1} & \\
 1 & & \bar{1} & & \bar{1} & & 1 \\
 & 0 & & 1 & & 0 & \\
 0 & 0 & 0 & 0 & 0 & 0 &
\end{matrix}
\qquad
\mathcal{V}_2 = \begin{matrix}
 & & 0 & & & \\
 & 0 & & 0 & & \\
 1 & & & 0 & & \\
 \bar{1} & \bar{1} & & 1 & 0 & \\
 \bar{1} & & \bar{1} & & 0 & \\
 1 & \bar{1} & \bar{1} & 1 & 0 & 0
\end{matrix}
$$

$$
\mathcal{V}_3 = \begin{matrix}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & & 1 & & \\
 0 & & 1 & \bar{1} & \bar{1} & & \\
 0 & & & \bar{1} & & \bar{1} & \\
 0 & 0 & 1 & \bar{1} & \bar{1} & 1 &
\end{matrix}\;.
\tag{13}
$$

The generalization to higher rank $su(r + 1)$ is straightforward. In section 3 we use another choice of indices on $\mathcal{V}$.

We are now in a position to generate all generalized BZ triangles associated with a given triple coupling. Once a single generalized triangle is found, the lattice of triangles associated with the triple coupling is spanned by adding integer linear combinations of the virtual triangles. The choice of initial triangle is not important. We denote the integer coefficients as *linear coefficients*. We emphasize that the Euclidean spaces spanned by the lattices are all of dimension H_r, i.e., the dimension is independent of the triple coupling and depends only on the rank of $su(r + 1)$.

Negative entries in (generalized) BZ triangles also appear in [5] . That work is mainly devoted to the construction of tensor-product generating functions. A new method is proposed based on elementary solutions to certain sets of linear equations related to the BZ triangles, and for $su(3)$ one of these solutions corresponds to a triangle with negative entries. The appearance of negative entries is expected to be a general feature for higher $su(N)$ as well. The elementary solutions are closely related to the so-called elementary couplings.

3. Polytopes, multiple sums and tensor product multiplicities

We shall now focus on the H_r-dimensional linear coefficient space and seek an algebraic description of the tensor product multiplicities. The latter are computed by counting true BZ triangles. Demanding that the entries of a *true* BZ triangle be non-negative, we obtain E_r inequalities the linear coefficients must satisfy. The inequalities depend on the choice of initial triangle, and they correspond to a polyhedral combinatorial expression for the multiplicities in linear coefficient space. The structure of the basis virtual triangles (cf (11), (12) and (13))

ensures that all linear coefficients have upper as well as lower bounds. The polyhedron is therefore bounded and such a polyhedron is called a polytope. It is easily seen to be convex.

In order to specify the polytope, we must find an initial triangle of weight (λ, μ, ν). It is convenient to break the symmetry among the three weights, and first look at the unique true triangle of weight $(\lambda, \mu, \lambda^+ + \mu^+)$. For *su*(3), this triangle is

$$
\begin{array}{ccccc}
& & \lambda_1 & & \\
& 0 & & \mu_1 & \\
& \lambda_2 & & \lambda_2 & \\
0 & & \mu_1 & 0 & \mu_2
\end{array}
\tag{14}
$$

and the generalization to $su(r + 1)$ is clear. Every highest weight ν in a coupling $\lambda \otimes \mu \otimes \nu$ satisfies

$$
\nu = \lambda^+ + \mu^+ - \sum_{i=1}^{r} n_i \alpha_i
\tag{15}
$$

with $n_i \in \mathbb{Z}_{\geqslant}$, where α_i is the ith simple root. The coefficients n_i are conveniently expressed using dual Dynkin labels. A weight λ can be written

$$
\lambda = \sum_{i=1}^{r} \lambda_i \Lambda^i = \sum_{i=1}^{r} \lambda^i \alpha_i^\vee
\tag{16}
$$

where $\{\Lambda^i\}$ and $\{\alpha_i^\vee\}$ are the sets of fundamental weights and simple co-roots, respectively. The λ^i are the dual Dynkin labels, while the ordinary Dynkin labels are the λ_i. For simply laced algebras, such as $su(N)$, α_i is identical to the co-root $\alpha_i^\vee$ (with standard normalization $\alpha^2 = 2$, for α a long root). Taking the scalar product of (15) with Λ^i therefore gives

$$
n_i = (\lambda^+)^i + (\mu^+)^i - \nu^i.
\tag{17}
$$

Generalized triangles of weight $(0, 0, \alpha_i)$ are also easily constructed. An *su*(3) example is

$$
\begin{array}{ccccc}
& & 1 & & \\
& \bar{1} & & 1 & \\
& 0 & & \bar{1} & \\
0 & & 0 & 0 & 0
\end{array}
\tag{18}
$$

of weight $(0, 0, \alpha_2)$. So, one can find a generalized triangle of weight (λ, μ, ν) by subtracting non-negative integer multiples of triangles of weight $(0, 0, \alpha_i)$, such as (18), from a triangle like (14), of weight $(\lambda, \mu, \lambda^+ + \mu^+)$.

The result for $su(r+1)$ is the following generalized BZ triangle associated with the triple product $\lambda \otimes \mu \otimes \nu$:

$$
\begin{array}{ccccccccccccccc}
& & & & & & & N'_r & & & & & & & \\
& & & & & & n_r & & N_r & & & & & & \\
& & & & & \lambda_2 & & & & N'_{r-1} & & & & & \\
& & & & 0 & & \mu_1 & & n_{r-1} & & N_{r-1} & & & & \\
& & & \lambda_3 & & & & \lambda_3 & & & & N'_{r-2} & & & \\
& & 0 & & \mu_1 & & 0 & & \mu_2 & & n_{r-2} & & N_{r-2} & & \\
& \iddots & & & & & \vdots & & \vdots & & & & \ddots & & \\
& \lambda_{r-2} & & & & & & & & & & N'_3 & & & \\
0 & & \mu_1 & & & & & & & & n_3 & & N_3 & & \\
\lambda_{r-1} & & & \lambda_{r-1} & & & \cdots & & & \lambda_{r-1} & & & & N'_2 & \\
0 & & \mu_1 & & 0 & & \mu_2 & & & 0 & & \mu_{r-2} & & n_2 & N_2 \\
\lambda_r & & & \lambda_r & & & \lambda_r & & & \lambda_r & & & \lambda_r & & N'_1 \\
0 & & \mu_1 & & 0 & & \mu_2 & & 0 & \mu_3 & \cdots & 0 & \mu_{r-2} & 0 & \mu_{r-1} & n_1 & N_1.
\end{array}
\tag{19}
$$

The entries n_i, N'_i N_i are defined by

$$
\begin{aligned}
n_i &= \lambda^{r-i+1} + \mu^{r-i+1} - \nu^i \\
N_i &= (1 - \delta_{i1})n_{i-1} - n_i + \mu_{r-i+1} \\
&= -\lambda^{r-i+1} + (1 - \delta_{i1})\lambda^{r-i+2} - (1 - \delta_{ir})\mu^{r-i} + \mu^{r-i+1} - (1 - \delta_{i1})\nu^{i-1} + \nu^i \\
N'_i &= \nu_i - N_i \\
&= \lambda^{r-i+1} - (1 - \delta_{i1})\lambda^{r-i+2} + (1 - \delta_{ir})\mu^{r-i} - \mu^{r-i+1} + \nu^i - (1 - \delta_{ir})\nu^{i+1} .
\end{aligned}
\tag{20}
$$

In order to describe the polytope explicitly, we need to label the virtual triangles. Our choice is to write them as $\mathcal{V}_{i,j}$ or $\mathcal{V}_l$ depending on where the associated hexagons are situated. The corresponding linear coefficients are denoted $d_{i,j}$ and η_l:

$$
\begin{array}{ccccccccccccccc}
& & & & & & & \star & & & & & & & \\
& & & & & & \star & & \star & & & & & & \\
& & & & & \star & & \eta_{r-1} & & \star & & & & & \\
& & & & \star & & \star & & \star & & \star & & & & \\
& & & \star & & d_{r-2,1} & & \star & & \eta_{r-2} & & \star & & & \\
& & \star & & \star & & \star & & \star & & \star & & \star & & \\
& \iddots & & & & & \vdots & & \vdots & & & & \ddots & & \\
& \star & & & & & & & & & & \star & & & \\
\star & & \star & & & & & & & & \star & & \star & & \\
\star & & d_{2,1} & & \star & & & \cdots & & & \star & & \eta_2 & & \star \\
\star & & \star & & \star & & \star & & & & \star & & \star & & \star & \star \\
\star & & d_{1,1} & & \star & & d_{1,2} & & \star & & \star & & d_{1,r-2} & & \star & \eta_1 & \star \\
\star & & \star & & \star & & \star & & \star & & \star & \cdots & \star & & \star & \star & \star & \star
\end{array}
\tag{21}
$$

Here a $\star$ indicates an unspecified entry while $d_{i,j}$ and η_l are the linear coefficients of the virtual triangles. They are depicted at the centres of the hexagons associated with the corresponding virtual triangles. We have chosen two different notations for the virtual triangles (and their associated linear coefficients) to reflect the positions of the corresponding hexagons in the asymmetric initial triangle (19).

Now, denoting the initial triangle $\mathcal{T}_0$, any triangle in the lattice of general triangles of weight (λ, μ, ν) may be written as

$$\mathcal{T} = \mathcal{T}_0 + \sum_{l=1}^{r-1} \eta_l \mathcal{V}_l + \sum_{\substack{i,j \geqslant 1}}^{i+j=r-1} d_{i,j} \mathcal{V}_{i,j}. \tag{22}$$

The associated polytope of interest is in the H_r-dimensional space spanned by $d_{i,j}$ and η_l. It is bounded by the inequalities requiring that all entries in $\mathcal{T}$ are non-negative (whereby $\mathcal{T}$ is ensured to be a pure BZ triangle). Hence, the position of the polytope depend on the initial triangle $\mathcal{T}_0$. Nevertheless, the *volume* of the polytope, the number of integer points bounded by the polytope, is *independent* of $\mathcal{T}_0$. By construction, this number is the tensor product multiplicity $T_{\lambda,\mu,\nu}$ of the triple coupling $\lambda \otimes \mu \otimes \nu$ to the singlet.

Using the explicit choice of initial triangle (19) and the basis of virtual triangles discussed above, it is simple to write down the inequalities defining the polytope. To illustrate, we list the three inequalities given by the three entries located furthest to the right on the bottom line of $\mathcal{T}$:

$$\mu_{r-1} + d_{1,r-2} - \eta_1 \geqslant 0 \qquad n_1 - \eta_1 \geqslant 0 \qquad N_1 + \eta_1 \geqslant 0. \tag{23}$$

A similar polyhedral combinatorial expression is discussed in [6]. Convex polytopes constructed there lie in the space of Gelfand–Tsetlin patterns (see, e.g., [6, 2]), while ours lie in spaces associated with BZ triangles. Hence, for $su(N)$ their polytopes are embedded in the Euclidean vector space $\mathbb{R}^{N(N+1)/2}$, while ours may be embedded in the smaller space $\mathbb{R}^{(N-2)(N-1)/2}$. A more universal method of constructing polyhedral combinatorial expressions for tensor product multiplicities, generalizing those of [6] and making sense for any simple Lie algebra, may be found in [7].

3.1. Explicit multiple sum formula

As already stated, our polyhedral expression differs from the one discussed in [6, 7]. Its structure allows us to extract an explicit multiple sum formula counting the integer points bounded by the polytope. The multiple sum is over the linear coefficients, so different orders of summation give a total of $H_r!$ possible representations of the polytope volume. For practical purposes, however, there are considerably fewer appropriate summation orders. Let us illustrate our procedure for choosing an appropriate order of summation by considering the following simple planar example.

Let a planar polytope be defined by the set of inequalities

$$\begin{matrix} 1 \leqslant x \leqslant 4 & \quad 8 \leqslant x + y \leqslant 14 \\ 6 \leqslant y & \quad 4 \leqslant y - x \leqslant 8. \end{matrix} \tag{24}$$

The volume or area $\mathcal{A}$ of the polytope (the number of integer points bounded by the inequalities) can be written in two ways:

$$\mathcal{A} = 16 = \sum_{x=1}^{4} \sum_{y=\max\{6,x+4,8-x\}}^{\min\{x+8,14-x\}} 1 = \sum_{y=6}^{11} \sum_{x=\max\{1,y-8,8-y\}}^{\min\{4,y-4,14-y\}} 1. \tag{25}$$

The second expression is slightly more difficult to write, since the upper limit 11 on y must be calculated from the intersection of faces (lines). Here the bounding lines $x + y = 14$ and $y - x = 8$ intersect at the point $(x, y) = (3, 11)$. This is a complication to avoid while writing our formula, since it will involve many sums. In the first expression the explicitly written lower limit $y = 6$ is redundant since the remaining two intersect at the point $(2, 6)$. However,

including redundant limits does not change the result, so we may choose to keep the limit $y = 6$.

It is clear that an important difference between the two orders of summation is that we have $1 \leqslant x \leqslant 4$, but only $6 \leqslant y$. For example, if (24) is supplemented with $y \leqslant 15$, then the upper limit 11 can be replaced simply by 15, and the formula is still valid. While it is true that the new formula contains single sums with lower limits greater than upper limits, this is a relatively small inconvenience. Those sums simply contribute 0.

So, while choosing an appropriate order of summation (over the summation variables η_l and $d_{i,j}$), it is crucial for any summation variable that all subsequent summation variables have upper as well as lower bounds parallel to (or independent of) the one under consideration.

This is a non-trivial consideration: the procedure does not apply to all polytopes. A trivial example is provided by (24) with the inequality $x \leqslant 4$ removed.

Fortunately, the simplifying procedure applies to our polytope. A simple inspection reveals that not all summation orders are appropriate, however. Nevertheless, in the general case and in accordance with our procedure, we may express the volume of the polytope as

$$T_{\lambda,\mu,\nu} = \left(\sum_{d_{1,1}}\right)\left(\sum_{d_{2,1}}\sum_{d_{1,2}}\right)\cdots\left(\sum_{d_{r-2,1}}\cdots\sum_{d_{1,r-2}}\right)\left(\sum_{\eta_{r-1}}\cdots\sum_{\eta_1}\right)1 \qquad (26)$$

where the summation variables are bounded according to

$$\max\{-N_1, d_{1,r-2}, -N_2' + \eta_2, -\mu_{r-2} + d_{1,r-2} - d_{2,r-3} + \eta_2\}$$
$$\leqslant \eta_1 \leqslant \min\{n_1, \mu_{r-1} + d_{1,r-2}, \lambda_r - d_{1,r-2} + \eta_2, n_2 + d_{1,r-2} - \eta_2, N_1', N_2 + \eta_2\}$$
$$\max\{d_{l-1,r-l} - d_{l-1,r-l-1} + d_{l,r-l-1}, -N_{l+1}' + \eta_{l+1},$$
$$\qquad -\mu_{r-l-1} + \eta_{l+1} - (1 - \delta_{l,r-2})d_{l+1,r-l-2} + d_{l,r-l-1}\}$$
$$\leqslant \eta_l \leqslant \min\{\lambda_{r-l+1} - d_{l,r-l-1} + d_{l-1,r-l} + \eta_{l+1},$$
$$\qquad n_{l+1} + d_{l,r-l-1} - \eta_{l+1}, N_{l+1} + \eta_{l+1}\} \qquad \text{for} \quad 2 \leqslant l \leqslant r-2$$
$$\max\{d_{r-2,1}, -N_r'\} \leqslant \eta_{r-1} \leqslant \min\{\lambda_2 + d_{r-2,1}, n_r, N_r\}$$
$$\max\{d_{1,j-1}, -\mu_{j-1} + d_{1,j-1} + d_{2,j-1} - (1 - \delta_{j,2})d_{2,j-2}\}$$
$$\leqslant d_{1,j} \leqslant \min\{\mu_j + d_{1,j-1}, \lambda_r - d_{1,j-1} + d_{2,j-1}\} \qquad \text{for} \quad 2 \leqslant j \leqslant r-2$$
$$\max\{d_{i,j-1} + d_{i-1,j} - d_{i-1,j-1}, -\mu_{j-1} + d_{i,j-1} + d_{i+1,j-1} - (1 - \delta_{j,2})d_{i+1,j-2}\}$$
$$\leqslant d_{i,j} \leqslant \lambda_{r-i+1} - d_{i,j-1} + d_{i+1,j-1} + d_{i-1,j} \qquad \text{for} \quad 2 \leqslant i, j, i+j \leqslant r-1$$
$$d_{i-1,1} \leqslant d_{i,1} \leqslant \lambda_{r-i+1} + d_{i-1,1} \qquad \text{for} \quad 2 \leqslant i \leqslant r-2$$
$$0 \leqslant d_{1,1} \leqslant \min\{\mu_1, \lambda_r\}. \qquad (27)$$

From (15) and (17) it follows that the weights are subject to the condition

$$\lambda^i + \mu^i + \nu^i \in \mathbb{Z}_{\geqslant} \qquad i = 1, \ldots, r \qquad (28)$$

ensuring the integer nature of the entries, and thus also the summation limits (27). The multiple sum formula (26) is our main new result. We now demonstrate its usefulness by considering an application and working out a few examples.

4. An application

It is of interest to know whether or not a coupling of a certain weight (λ, μ, ν) exists, without having to work out the tensor product multiplicity. Based on our multiple sum formula (26) and (27), one may derive a set of inequalities in the dual and ordinary Dynkin labels of the

three weights, determining when the associated tensor product multiplicity is non-vanishing. To illustrate the method, we discuss the inequalities for *su*(3) and *su*(4) and outline their derivation. In principle, it is possible to repeat the procedure for higher rank, but even for *su*(4) the derivation is very cumbersome. We believe that similar results exist for all simple Lie algebras, and hope to report more general results later.

The dimension of the linear coefficient space for *su*(3) is one, so the tensor product multiplicity may be represented by a single sum:

$$T_{\lambda,\mu,\nu} = \sum_{\eta=\max\{0,\lambda^2+\mu^1-\mu^2-\nu^1,-\lambda^1+\lambda^2+\mu^1-\nu^2\}}^{\min\{\mu_1,\lambda_2,\lambda^2+\mu^2-\nu^1,\lambda^1+\mu^1-\nu^2,-\lambda^1+\lambda^2+\mu^1-\nu^1+\nu^2,\lambda^2+\mu^1-\mu^2+\nu^1-\nu^2\}} 1. \tag{29}$$

The weights are subject to the integer constraint (28). Note that the summation limits are not symmetric in the weights. This is simply because we have chosen an asymmetric initial triangle. The summation (29) is non-vanishing if and only if the upper limit is greater than or equal to the lower limit. This condition yields $6 \times 3 = 18$ inequalities:

$$0 \leqslant \lambda_i, \mu_i, \nu_i \qquad \text{for} \quad i = 1, 2$$
$$\max\{\lambda^1 - \lambda^2 + \mu^1 - \mu^2, -\lambda^1 + \mu^2, \lambda^2 - \mu^1\} \leqslant \nu^1 \leqslant \lambda^2 + \mu^2$$
$$\max\{-\lambda^1 + \lambda^2 - \mu^1 + \mu^2, \lambda^1 - \mu^2, -\lambda^2 + \mu^1\} \leqslant \nu^2 \leqslant \lambda^1 + \mu^1 \tag{30}$$
$$\max\{-\lambda^2 - \mu^1 + \mu^2, -\lambda^1 + \lambda^2 - \mu^2\} \leqslant \nu^1 - \nu^2$$
$$\leqslant \min\{\lambda^1 - \mu^1 + \mu^2, \lambda^1 + \lambda^2 + \mu^1\}.$$

Expressing the inequalities in terms of ordinary Dynkin labels, one should bear in mind that the summation variable increases in steps of one while the quadratic-form matrix involves a factor of 1/3. A similar factor 1/*N* is present for higher rank *su*(*N*).

In the case of *su*(4) the tensor product multiplicity may be written as a triple sum

$$T_{\lambda,\mu,\nu} = \sum_{d=0}^{\min\{\mu_1,\lambda_3\}} \sum_{\eta_2=\max\{d,-\lambda^1+\lambda^2+\mu^1-\nu^3\}}^{\min\{\lambda_2+d,\lambda^1+\mu^1-\nu^3,-\lambda^1+\lambda^2+\mu^1-\nu^2+\nu^3\}}$$
$$\times \sum_{\eta_1=\max\{\lambda^3+\mu^2-\mu^3-\nu^1,d,-\lambda^2+\lambda^3-\mu^1+\mu^2-\nu^2+\nu^3+\eta_2,-\mu_1+d+\eta_2\}}^{\min\{\lambda^3+\mu^3-\nu^1,\mu_2+d,\lambda_3-d+\eta_2,\lambda^2+\mu^2-\nu^2+d-\eta_2,\lambda^3+\mu^2-\mu^3+\nu^1-\nu^2,-\lambda^2+\lambda^3-\mu^1+\mu^2-\nu^1+\nu^2+\eta_2\}} 1. \tag{31}$$

The weights are subject to the integer constraint (28). For a multiple sum like this, the inequalities are obtained by first considering the interior summation over η_1, leading to $6 \times 4 = 24$ inequalities which may depend on the remaining two summation variables: four of them depend only on the weights, six depend on d but not η_2, while 14 depend on η_2. Treating the latter in the same way as the upper and lower bounds on the η_2-summation, we obtain a total of $13 \times 6 = 78$ inequalities from the η_2-consideration. Repeating the procedure for d leads to a total of $54 \times 10 = 540$ inequalities in addition to the ones already derived from the η_1- and η_2-considerations. This huge set of inequalities may be reduced considerably and we find the following constraints on the Dynkin labels (expressed in terms of dual as well as ordinary ones)

$$0 \leqslant \lambda_i, \mu_i, \nu_i \qquad \text{for} \quad i = 1, 2, 3$$
$$\max\{\lambda^3 - \mu^1, \lambda^3 - \lambda_3 - \mu^1 + \mu_1, -\lambda^1 + \mu^3, -\lambda^1 + \lambda_1 + \mu^3 - \mu_3\} \leqslant \nu^1 \leqslant \lambda^3 + \mu^3$$
$$\max\{|\lambda^2 - \mu^2|, |\lambda^2 - \lambda_2 - \mu^2 + \mu_2|, |\lambda^1 - \lambda^3 + \mu^1 - \mu^3|\} \leqslant \nu^2 \leqslant \lambda^2 + \mu^2$$
$$\max\{\lambda^1 - \mu^3, \lambda^1 - \lambda_1 - \mu^3 + \mu_3, -\lambda^3 + \mu^1, -\lambda^3 + \lambda_3 + \mu^1 - \mu_1\} \leqslant \nu^3 \leqslant \lambda^1 + \mu^1$$

$$\max\{\lambda^2 - \lambda^3 - \mu^1, -\lambda^1 + \mu^2 - \mu^3, \lambda^1 - \lambda^2 + \mu^1 - \mu^2\}$$
$$\leqslant \nu^1 - \nu_1$$
$$\leqslant \min\{\lambda^2 - \lambda^3 + \mu^3, \lambda^3 + \mu^2 - \mu^3\}$$
$$\max\{-\lambda^1 + \lambda^3 + \mu_2 - \mu^2, \lambda_2 - \lambda^2 - \mu^1 + \mu^3, \lambda^1 - \lambda^3 + \mu_2 - \mu^2,$$
$$\lambda_2 - \lambda^2 + \mu^1 - \mu^3, -\lambda^2 + \mu^2 - \mu_2, \lambda^2 - \lambda_2 - \mu^2\}$$
$$\leqslant \nu^2 - \nu_2 \tag{32}$$
$$\leqslant \min\{\lambda^2 - \lambda_2 + \mu^2, \lambda^2 + \mu^2 - \mu_2\}$$
$$\max\{\lambda^2 - \lambda^1 - \mu^3, -\lambda^3 + \mu^2 - \mu^1, \lambda^3 - \lambda^2 + \mu^3 - \mu^2\}$$
$$\leqslant \nu^3 - \nu_3$$
$$\leqslant \min\{\lambda^2 - \lambda^1 + \mu^1, \lambda^1 + \mu^2 - \mu^1\}$$
$$\max\{-\lambda^1 + \lambda^3 - \mu^2, -\lambda^2 - \mu^1 + \mu^3, \lambda_2 - \lambda^2 + \mu_2 - \mu^2\}$$
$$\leqslant \nu^1 - \nu^3$$
$$\leqslant \min\{-\lambda^1 + \lambda^3 + \mu^2, \lambda^2 - \mu^1 + \mu^3, \lambda^2 - \lambda_2 + \mu^2 - \mu_2\}.$$

A recent discussion [8] includes a brief review of the problem of determining when a tensor product multiplicity is non-vanishing. The focus there is on $gl(N)$ (and therefore also on $su(N)$) and the results are provided in the form of polyhedral combinatorial expressions. Our prescription described above is in general more explicit than the results obtained previously. For lower rank $su(N)$ though, the explicit inequalities may be obtained using various alternative approaches.

4.1. A refinement

Here we shall indicate how one may derive sets of inequalities determining when a tensor product multiplicity is greater than a given non-negative integer K

$$T_{\lambda,\mu,\nu} > K. \tag{33}$$

The case $K = 0$ was discussed above.

Our approach is straightforward since the problem translates into studying when a convex polytope has a (discretized) volume bigger than K. To illustrate, let us consider $su(3)$. In this case the volume is expressed as a single sum (29), so (33) is equivalent to requiring

$$\min\{\mu_1, \lambda_2, \lambda^2 + \mu^2 - \nu^1, \lambda^1 + \mu^1 - \nu^2, -\lambda^1 + \lambda^2 + \mu^1 - \nu^1 + \nu^2, \lambda^2 + \mu^1 - \mu^2 + \nu^1 - \nu^2\}$$
$$- \max\{0, \lambda^2 + \mu^1 - \mu^2 - \nu^1, -\lambda^1 + \lambda^2 + \mu^1 - \nu^2\} \geqslant K. \tag{34}$$

This leaves us the following 18 inequalities refining (30)

$$K \leqslant \lambda_i, \mu_i, \nu_i \quad \text{for} \quad i = 1, 2$$
$$\max\{\lambda^1 - \lambda^2 + \mu^1 - \mu^2, -\lambda^1 + \mu^2, \lambda^2 - \mu^1\} + K \leqslant \nu^1 \leqslant \lambda^2 + \mu^2 - K$$
$$\max\{-\lambda^1 + \lambda^2 - \mu^1 + \mu^2, \lambda^1 - \mu^2, -\lambda^2 + \mu^1\} + K \leqslant \nu^2 \leqslant \lambda^1 + \mu^1 - K \tag{35}$$
$$\max\{-\lambda^2 - \mu^1 + \mu^2, -\lambda^1 + \lambda^2 - \mu^2\} + K \leqslant \nu^1 - \nu^2$$
$$\leqslant \min\{\lambda^1 - \mu^1 + \mu^2 - \lambda^1 + \lambda^2 + \mu^1\} - K.$$

To the best of our knowlededge, this is a new result.

In the case of $su(4)$ the situation is already much more complicated. That is because the polytope is three dimensional, and we cannot immediately use the triple-sum formula (31). We recall that our simplifying method for obtaining an appropriate order of summation, may

include redundant summations contributing zero to the final expression. We may therefore lose crucial information if we only consider the multiple sum formula. The remedy is to consider the original polytope, and require the defining faces to embrace a volume of at least the desired value. We would then be led to consider three-dimensional partitions of $K + 1$, which is beyond the scope of the present paper. For lower values of K the problem is straightforward, though.

5. Conclusion

By virtue of virtual BZ triangles we have obtained a polyhedral combinatorial expression for the $su(N)$ tensor product multiplicities, different from those discussed in [6, 7]. The main merit of our expression is that it admits a simple measurement of the convex polytope volume in terms of a multiple sum formula. The latter is then a new and explicit way of expressing the tensor product multiplicities of $su(N)$.

As an application, one may derive explicit bounds on a triple of weights determining when the associated coupling to the singlet exists. To illustrate, the bounds were written for $su(3)$ and $su(4)$. Also included was a brief discussion on how to generalize this to bounds describing $T_{\lambda,\mu,\nu} > K$.

We believe that our multiple sum representation of the tensor product multiplicities provides a significant computational improvement over previous (combinatorial) results. In particular, it is expected to lead to considerable simplifications when implemented in computer programs.

It is our hope that our results may find applications to the computation of fusion rules in conformal field theory with affine Lie group symmetry, the so-called WZW theories. Since tensor product multiplicities correspond to the infinite-level limit of fusion multiplicities, it is helpful to have simple descriptions of the former in order to understand the latter.

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